may be noted that if Nowinski had used an energy approach, his results might tend to agree with those of the writer.

Thus it appears that slightly different formulations of the "same" problem can lead to significantly different results. Additional experiments and refinements of the analysis are in progress at Graduate Aeronautical Laboratories, California Institute of Technology, and a full report will be issued on completion of the investigations. At this writing, however, it is safe to say that 1) the constraint on v should be satisfied in all problems involving the deflections of complete cylindrical shells, and 2) the complementary solutions of the compatibility equation (3) warrant further discussion.

References

[1] Chu, H. N., "Influence of large amplitudes on flexural vibrations of a thin circular cylindrical shell," J. Aerospace Sci. **28**, 602–609 (1961).

[2] Nowinski, J., "Nonlinear transverse vibrations of orthotropic cylindrical shells," AIAA J. **1**, 617–620 (1963).

[3] Reissner, E., "Nonlinear effects in vibrations of cylindrical shells," Ramo-Wooldridge Corp. Rept. AM5-6 (September 1955).

[4] Cummings, B. E., "Large amplitude vibration and response of curved panels," Inst. for Defense Analyses Res. Paper P-13 Washington, D. C. (March 1963).

[5] von Kármán, T. and Tsien, H. S., "The buckling of thin cylindrical shells under axial compression," J. Aeronaut. Sci. **8**, 303–312 (June 1941).

Optimum Launching to Hyperbolic Orbit by Two Impulses

Stephen Lubard*

Polytechnic Institute of Brooklyn, Brooklyn, N. Y.

In this analysis the optimum launching of an object into a nonintersecting hyperbolic orbit is investigated. The effects of rotation and atmospheric drag were not included. The conclusions of this analysis can also be applied to the optimum landing on a planet from a hyperbolic orbit. Only the planar transfer is investigated, since this is always more economical than the nonplanar transfer.[1]

For the case of equal specific impulses, the problem of optimizing the launching is equivalent to minimizing the total velocity change I,

$$I = |\mathbf{V}_0| + |\mathbf{u}_2 - \mathbf{u}_1| \tag{1}$$

subject to the constraints of conservation of angular momentum and energy:

$$h_1 = V_0 \cos\omega_0 = u_1\rho \cos(\omega_2 + z) \tag{2}$$

$$h_2 = u_2\rho \cos\omega_2 \tag{3}$$

$$E_1 = \tfrac{1}{2}V_0^2 - 1 = \tfrac{1}{2}u_1^2 - 1/\rho \tag{4}$$

$$E_2 = \tfrac{1}{2}u_2^2 - 1/\rho \tag{5}$$

where subscripts 0, 1, and 2 refer to the launching, transfer orbit, and final hyperbola, as shown in Fig. 1. The radius of the planet and the gravitational acceleration have been equated to unity. With the four constraint equations we must optimize I with respect to three variables. Following Ref. 2, I will be optimized with respect to z, ω_0, and u_2.

Received August 28, 1963. The author wishes to acknowledge Lu Ting for his guidance in preparing this analysis.

* Student, participant in the National Science Foundation Science Education Program.

Substituting Eq. (4) into Eq. (1), one gets

$$I = \left[2 + u_1^2 - \frac{2}{\rho}\right]^{1/2} + [u_1^2 + u_2^2 - 2u_1u_2 (\cos z)]^{1/2} \tag{6}$$

which, for a given ρ and u_1, will be a minimum if $z = 0$. Therefore, the transfer trajectory should be tangent to the final hyperbola when firing the second impulse.

Equation (1) with $z = 0$ becomes

$$I = V_0 + |u_2 - u_1| \tag{7}$$

In order to remove the absolute value sign, we must show $u_2 \geq u_1$. Equations (3–5) yield

$$h_2^2 - 2E_2 - 2 = u_2^2(\rho^2 \cos^2\omega_2 - 1) - 2[1 - (1/\rho)] = [p_2 + (e_2 - 1)][p_2 - (e_2 + 1)]/p_2 \geq 0 \tag{8}$$

since the eccentricity e_2 and the perigee $p_2/(e_2 + 1)$ of the final hyperbola cannot be less than unity.

Similarly,

$$h_1 - 2E_1 - 2 = u_1^2(\rho^2 \cos^2\omega_2 - 1) - 2[1 - (1/\rho)] = [p_1 + (e_1 - 1)][p_1 - (e_1 + 1)]/p_1 \leq 0 \tag{9}$$

since the perigee $p_1/(e_1 + 1)$ of the transfer orbit cannot be greater than unity.

It follows that $u_2 \geq u_1$. Substituting Eq. (2) with $z = 0$ into Eq. (7) yields

$$I = \frac{u_1\rho \cos\omega_2}{\cos\omega_0} + (u_2 - u_1) \tag{10}$$

For a given u_2 and ρ we may optimize I with respect to ω_0 by taking the first derivative,

$$\frac{dI}{d\omega_0} = \left[\frac{u_1\rho \cos\omega_2}{\cos\omega_0(\rho \cos\omega_2 + \cos\omega_0)}\right] \sin\omega_0 \tag{11}$$

Equation (11) equals zero if, and only if, $\omega_0 = 0$ for $0 \leq \omega_0 \leq \pi/2$. Since $(d^2I/d\omega_0^2) > 0$ at $\omega_0 = 0$, I is a minimum for a tangential launching ($\omega_0 = 0$).

I must now be minimized with respect to u_2. Substituting from Eqs. (2–5), with $z = \omega_0 = 0$ into Eq. (10), one obtains

$$I = h_2\left[\frac{2 + 2E_2 - u_2^2}{h_2^2 - u_2^2}\right]^{1/2} + u_2\left[1 - \left(\frac{2 + 2E_2 - u_2^2}{h_2^2 - u_2^2}\right)^{1/2}\right] \tag{12}$$

Differentiating (12), one gets

$$\frac{dI}{du_2} = F(h_2 - u_2)[h_2(2 + 2E_2 - u_2^2)^{1/2} - u_2(h_2^2 - u_2^2)^{1/2}]$$

where

$$F = \frac{(h_2^2 - u_2^2)^{1/2} - (2 + 2E_2 - u_2^2)^{1/2}}{(h_2^2 - u_2^2)^{3/2}(2 + 2E_2 - u_2^2)^{1/2}}$$

From Eq. (8) we know that $F > 0$. Therefore, $dI/du_2 = 0$ if,

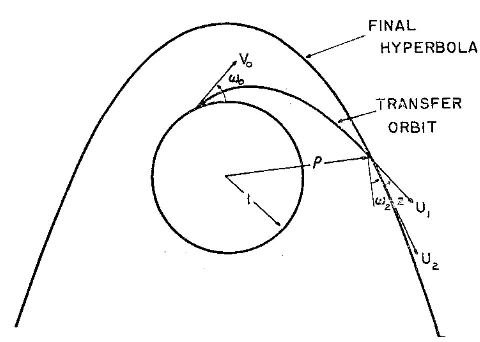

Fig. 1 Launching to hyperbolic orbit.

and only if, the term in the bracket vanishes. Evaluating the second derivative at $dI/du_2 = 0$ yields

$$\frac{d^2I}{du_2^2} = -F(h_2 - u_2)\left\{u_2\left[\frac{h_2}{(2 + 2E_2 - u_2^2)^{1/2}} - \frac{u_2}{(h_2^2 - u_2^2)^{1/2}}\right] + (h_2^2 - u_2^2)^{1/2}\right\}$$

We conclude that $(d^2I/du_2^2) < 0$, and if any extreme of I exists, it must be a maximum. Therefore, the minimum value of I occurs at the extremes of u_2, either at the perigee or at infinity.

The value of I at infinity (I_∞) must be compared with the value of I at the perigee (I_p) to see which is less.

Using the substitutions $a = (e_2^2 - 1)^{1/2}/p_2$ and $b = (e_2 + 1)/p_2$, where $0 < a < b < 1$, one obtains from Eq. (12),

$$I_p = (2)^{1/2}\left[\frac{1 - b}{(1 + b)^{1/2}} + \frac{b(b)^{1/2}}{(b^2 - a^2)^{1/2}}\right] \tag{13}$$

and

$$I_\infty = (2)^{1/2}\left[\left(\frac{1 - a}{1 + a}\right)^{1/2} + \frac{a(b)^{1/2}}{(b^2 - a^2)^{1/2}}\right] \tag{14}$$

We see from Eqs. (13) and (14) that as $a \to 0$, $I_p > I_\infty$, and as $a \to b$, $I_\infty > I_p$. Therefore, there is some value of a as a function of b, such that $I_p = I_\infty$. Setting (13) equal to (14), one obtains

$$-\frac{(1 - b)^{1/2}(1 - b)^{1/2}}{(1 + b)^{1/2}} + \left(\frac{1 - a}{1 + a}\right)^{1/2} = (b)^{1/2}\left(\frac{b - a}{b + a}\right)^{1/2}$$

Using the substitutions $x^2 = (1 - a)/(1 + a)$ and $y^2 = (1 - b)/(1 + b)$, we obtain a fourth degree equation in x, for which the only real positive solution is $x = [-b + (b^2 + b + 1)^{1/2}]/(1 + b)^{1/2}$; hence, $a = [b/(b^2 + b + 1)^{1/2}]$ for $I_p = I_\infty$.

Case 1. If $a > b/(b^2 + b + 1)^{1/2}$, then the optimum transfer occurs at the perigee and our problem is completely solved.

Case 2. If $a < b/(b^2 + b + 1)^{1/2}$, then the optimum transfer would be at infinity. For this case we are interested in the radial distance ρ^* for which the total impulse (I_ρ^*) of transfer is equal to the total impulse of transfer at the perigee (I_p).

From Eq. (12) we obtain I_ρ^*:

$$\frac{1}{2^{1/2}}I_\rho^* = \left[\frac{k - v}{k + v}\right]^{1/2}[1 + k^2a^2 - v^2]^{1/2} + v$$

where $v^2 = [1/\rho^* + ba^2/(b^2 - a^2)]$ and $k^2 = b/(b^2 - a^2)$.

The equation, $I_\rho^* = I_p$, becomes

$$v^2\left(2k - \frac{2}{2^{1/2}}I_p\right) + v\left(\frac{I_p^2}{2} - \frac{2}{2^{1/2}}I_pk + k^2a^2 + 1\right) + \left(\frac{kI_p}{2^{1/2}} - k - k^3a^2\right) = 0$$

The factor $v - k$ has been omitted for $v \leq bk < k$.

Since $1/\rho^* = b$ or $v = bk$ must be a solution, we obtain the other solution as $v = 1/(1 + b)^{1/2}$, which gives

$$\rho^* = \frac{(1 + b)(b^2 - a^2)}{b^2 - a^2(1 + b + b^2)}$$

Therefore, a transfer to the hyperbola at any $\rho > \rho^*$ requires less total impulse than a transfer at the perigee.

Another interesting quantity is the value of a as a function of b, such that a transfer at any point on the hyperbola is more economical than a transfer at the perigee. This is found by setting ρ^* equal to the perigee ($1/b$) and obtaining $a = b(1 - b - b^2)^{1/2}$. Therefore, if $a \leq b(1 - b - b^2)^{1/2}$, then a tangential transfer to any point on the hyperbola is more economical than a tangential transfer to the perigee.

Conclusion

It was found that the total impulse required is a minimum if the launching is done in the following manner:

1) Tangential launching from the planet.

2) The transfer trajectory is tangent to the hyperbolic orbit at the firing of the second impulse.

3a) For $a > [b/(b^2 + b + 1)^{1/2}]$, the second impulse should be fired at the perigee, where a denotes the ratio of the radius of the planet to the radial distance to the asymptote of the hyperbolic orbit, and b denotes the ratio of the radius of the planet to the radical distance to the perigee of the hyperbolic orbit.

3b) For $a < [b/(b^2 + b + 1)^{1/2}]$) the second impulse should be fired at infinity. For this case, if the second impulse is fired at any radial distance $\rho > [(1 + b)(b^2 - a^2)/b^2 - a^2(1 + b + b^2)]$, the transfer is more economical than if the second impulse is fired at the perigee. In addition, if $a \leq b(1 - b - b^2)^{1/2}$, then a transfer at any point on the final hyperbola is more economical than a transfer at the perigee.

Since we have a tangential launching, the value of the total impulse for a transfer from a circular to a hyperbolic orbit (I_{tr}) is related to that of the launching (I_{LN}) by the equality $I_{tr} = I_{LN} - 1$. Therefore, min I_{tr} = min $I_{LN} - 1$. This indicates that the results of his analysis can also be applied to the transfer from a circular to a nonintersecting hyperbolic orbit.

References

[1] Ting, L., "Optimum transfer by impulses," ARS J. 30, 1013–1018 (1960).

[2] Barrére, M., Jaumotte, A., Fraeijs de Veubeke, B., and Vandenkerckhove, J., *Rocket Propulsion* (Elsevier Publishing Co., New York, 1960), pp. 724–727.

Supersonic Transport Climb Path Optimization Including a Constraint on Sonic Boom Intensity

MICHAEL FALCO*

Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

The ability to compute minimum fuel acceleration-climb paths subject to prescribed sonic boom intensity limits for a representative supersonic transport design is demonstrated with a particular gradient penalty function technique. Numerical studies of these paths indicate a significant loss in range performance of approximately 500 statute miles when a conservative sonic boom intensity limit of 2.25 lb/ft² is observed.

Introduction

THE optimization of acceleration-climb paths for minimum fuel performance for the supersonic transport is regarded as a critical economic factor in the operation of these vehicles. Adherence to fuel-optimal paths however, intro-

Received September 3, 1963. This study was partially supported by the Air Force Office of Scientific Research under Contract No. AF29(600)-2671 and AF49(638)-1207. The author wishes to acknowledge the assistance of Allen Schneider for his preparation of the aerodynamic, propulsive, and sonic boom data for the representative vehicle considered, and of Joseph Lindorfer for his programming efforts.

* Research Engineer, Research Department. Member AIAA.